Midterm Exam (80pts)
Math 214 Section Q1 Winter 2010

Your name:________________________ ID#:______________

1.(30 pts) Test the series for convergence or divergence

a) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^n + 5^n}$

Solution.

Comparison test.

$$\frac{4^n}{3^n + 5^n} \le \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n .$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ is a geometric series with $r = 4/5$. Since $|r| < 1$, it is convergent.

Therefore, by the comparison test, the original series is also convergent.

Other possible tests: root test, limit comparison test, ratio test.

b) $\displaystyle\sum_{n=1}^{\infty} \left[\sin\left(\frac{\pi n^2 + n}{6n^2 + 3}\right)\right]^n$

Solution.

Root test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sin\left(\frac{\pi n^2 + n}{6n^2 + 3}\right) = \sin\frac{\pi}{6} = \frac{1}{2} < 1.$$

Therefore, the series is convergent by the root test.

c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution.

Integral test. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. f is clearly positive, continuous. f is also decreasing since the denominator is a product of increasing functions. So, all the assumptions of the integral test are satisfied.

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \left[u = \ln x,\ du = \frac{dx}{x}\right] = \int_{\ln 2}^{\infty} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{\ln 2}^{\infty} = \infty.$$

Thus, the integral is divergent. Therefore, by the integral test the series is also divergent.

2.(10 pts) Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{\sqrt{n}\, 5^n}$.

Solution. Test for absolute convergence using the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{\sqrt{n+1}\, 5^{n+1}} \frac{\sqrt{n}\, 5^n}{|x-4|^n} = \lim_{n\to\infty} \frac{|x-4|}{5} \sqrt{\frac{n}{n+1}}$$

$$= \frac{|x-4|}{5} < 1,$$

$$|x-4| < 5.$$

Thus, the radius of convergence is $R = 5$.

$$-5 < x - 4 < 5,$$

$$-1 < x < 9.$$

Now test the endpoints.

$$x = -1 \Rightarrow \qquad \sum_{n=1}^{\infty} \frac{(-1)^n (-5)^n}{\sqrt{n}\, 5^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a p-series with $p = 1/2 < 1$. Thus divergent.

$$x = 9 \Rightarrow \qquad \sum_{n=1}^{\infty} \frac{(-1)^n (5)^n}{\sqrt{n}\, 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This is an alternating series. $u_n = \frac{1}{\sqrt{n}}$ is a decreasing sequence, whose limit is zero. Therefore the latter series is convergent.

So, the interval of convergence of the original power series is $(-1, 9]$.

3.(10 pts) Find a power series representation for the function $f(x) = \frac{x^2}{(1+2x^2)^2}$.

Solution. We will use the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Differentiating both sides we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Replace x by $-2x^2$.

$$\frac{1}{(1+2x^2)^2} = \sum_{n=1}^{\infty} n(-2x^2)^{n-1} = \sum_{n=1}^{\infty} n(-1)^{n-1} 2^{n-1} x^{2n-2}.$$

Multiply both sides by x^2.

$$\frac{x^2}{(1+2x^2)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1} 2^{n-1} x^{2n}.$$

4.(10 pts) Find the slope of the tangent line to the polar curve $r = \sin 2\theta$ at the point where $\theta = \frac{\pi}{6}$.

Solution.

Using the relation between polar and Cartesian coordinates, we have

$$x = \sin 2\theta \cos\theta,$$

$$y = \sin 2\theta \sin\theta.$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta \sin\theta + \sin 2\theta \cos\theta}{2\cos 2\theta \cos\theta - \sin 2\theta \sin\theta}.$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{3}\sin\frac{\pi}{6} + \sin\frac{\pi}{3}\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}} = \frac{2\frac{1}{2}\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}}{2\frac{1}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\frac{1}{2}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}.$$

5.(10 pts) Find the area of the region enclosed by the cardioid $r = 1 + \cos\theta$.

Solution.

$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta = \frac{1}{2}(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\Big|_0^{2\pi}$$

$$= \frac{3\pi}{2}.$$

6.(10 pts) Find the area of the surface generated by revolving the polar curve $r = \cos\theta + \sin\theta$, $0 \le \theta \le \pi/2$, about the x-axis.

Solution.

$$S = \int_0^{\pi/2} 2\pi(\cos\theta + \sin\theta)\sin\theta\sqrt{(\cos\theta+\sin\theta)^2 + (-\sin\theta+\cos\theta)^2}\ d\theta$$

$$= 2\pi\int_0^{\pi/2} (\cos\theta\sin\theta + \sin^2\theta)$$
$$\times \sqrt{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta + \sin^2\theta - 2\cos\theta\sin\theta + \cos^2\theta}\ d\theta$$

$$= 2\pi\int_0^{\pi/2} (\cos\theta\sin\theta + \sin^2\theta)\sqrt{2}\ d\theta$$

$$= 2\sqrt{2}\pi\int_0^{\pi/2} (\frac{1}{2}\sin 2\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta)\ d\theta$$

$$= 2\sqrt{2}\pi(-\frac{1}{4}\cos 2\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)\Big|_0^{\pi/2}$$

$$= 2\sqrt{2}\pi(\frac{1}{4} + \frac{1}{4} + \frac{\pi}{4})$$

$$= \sqrt{2}\pi(1 + \frac{\pi}{2}).$$